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Consistency conditions for brane worlds in arbitrary dimensions

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Abstract: We consider "brane world sum rules" for compactifications involving an arbitrary number of spacetime dimensions. One of the most striking results derived from such consistency conditions is the necessity for negative tension branes to appear in five-dimensional scenarios. We show how this result is easily evaded for brane world models with more than five dimensions. As an example, we consider a novel realization of the Randall-Sundrum scenario in six dimensions involving only positive tension branes.

Keywords: Field Theories in Higher Dimensions, Extra Large Dimensions.

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Contents

1. Introduction

Brane world scenarios have captured the imagination of high energy theorists because they provide new mechanisms for resolving many problems in particle physics which have long resisted solution. In particular, this framework offers new explanations for the small ratio between the symmetry-breaking scale of electroweak physics, and the Planck scale of quantum gravity. One possibility [1, 2] is that the observed four-dimensional Planck scale is a derived quantity determined by the true fundamental scale, which may be as low as 1 TeV, and the compactification scale, which may be as large as a fraction of a millimeter. Another alternative [3, 4] is that this small ratio of energy scales appears as a gravitational redshift in a warped compactification of a theory with a single fundamental scale.

In these scenarios with large extra dimensions, the distance scales involved in compactification are much larger than that set by quantum gravity. Therefore, in constructing such models, one must take seriously the question of whether the space-time geometry, along with the accompanying background fields and branes, solves the classical Einstein equations in higher dimensions. In ref. [5], the authors showed how, from Einstein's equations, one can derive a set of consistency conditions or "sum rules" that must be satisfied by any such model.[1] One of their most striking results was to show that for a very broad class of models, e.g., those given in refs. [3, 6, 7], a

[1]Similar analyses appeared earlier, in studying warped compactifications without branes [11] and certain five-dimensional brane world scenarios [12].

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consistent compactification demands the inclusion of negative tension branes. This is a rather disappointing conclusion as negative tension branes are inherently unstable objects, although these instabilities may be avoided by certain constructions in string theory — see, e.g., ref. [8].

However, this result on the necessity of negative tension branes cannot be completely general as it is straightforward to construct consistent compactifications in six or higher dimensions which only involve positive tension branes (as will be discussed in detail below). Of course, there is no mystery here as a key ingredient in ref. [5] was that the analysis was limited to theories in *five dimensions!*

The purpose of this paper is to extend the consistency conditions to brane world scenarios in an arbitrary number of spacetime dimensions. We present these calculations in section 2, where we also demonstrate how contributions from, e.g., non-vanishing curvature on the internal space allow consistent compactifications to be constructed with only positive tension branes. In section 3, we examine how the "sum rules" are satisfied in detail in some six-dimensional models. In particular, we consider an interesting warped compactification based on the AdS soliton [9] which realizes the Randall-Sundrum hierarchy mechanism with only positive tension branes. A very similar brane world scenario was considered earlier in ref. [10]. We close in section 4 with a discussion of our results and some concluding remarks.

While this paper was in preparation, ref. [14] appeared which presents similar constraints for the case of $D = 6$ and $p = 3$.

2. Consistency conditions

Following the approach of ref. [5], we use Einstein's equations to derive some general formulae for the consistency of brane world models with a compact internal space. The full spacetime will be D-dimensional. The metric will have a warped product ansatz:

$$ds^2 = G_{MN}(X)dX^M dX^N = g_{mn}(y)dy^m dy^n + W^2(y)g_{\mu\nu}(x)dx^\mu dx^\nu\,, \tag{2.1}$$

where X^M denote coordinates on the full D-dimensional space, the $p+1$ coordinates x^μ denote the uncompactified directions in the spacetime, and the remaining $D-p-1$ coordinates y^m specify the compact internal space. As some examples: for $D = 5$, $p = 3$ and $W(y) = e^{-2k|y|}$, the above metric corresponds to that studied by Randall and Sundrum [3, 4] where two copies of a portion of five-dimensional anti-de Sitter space (AdS_5) are pasted together along three-brane boundaries. When $p = 3$ and $W = 1$ the metric is factorizable as considered in ref. [1]. Note that in the latter case, the internal compact space is usually considered to be a $(D-p-1)$-dimensional torus. We will allow for some generalizations in section 3.

In keeping with the standard nomenclature, we will refer to x^μ as the brane coordinates. However, in the following we will allow for the possibility that the model includes q-branes with $q > p$ and which are, therefore, extended in some of the internal space directions as well. The latter, of course, arises in many interesting string theoretic models [13]. For a D-dimensional model, one could consider q-branes with q as large as $D-1$, which would then be extended in all of the spacetime dimensions. In our analysis below, the net effect of such a space-filling brane would be to modify the cosmological constant, and so generally we will only discuss q-branes with $p \leq q \leq D-2$.

The components of the Ricci tensor in the full spacetime are related to their lower dimensional counterparts by

$$R_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{g_{\mu\nu}}{p+1}\frac{1}{W^{p-1}}\nabla^2 W^{p+1}\,, \tag{2.2}$$

$$R_{mn} = \tilde{\mathcal{R}}_{mn} - \frac{p+1}{W}\nabla_m\nabla_n W\,, \tag{2.3}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor derived from $g_{\mu\nu}$ (independent of the warp factor) and $\tilde{\mathcal{R}}_{mn}$ is the Ricci tensor derived from the internal metric g_{mn}. Here ∇_m and ∇^2 are respectively the covariant derivative and the covariant laplacian with respect to this internal metric. The three relevant Ricci scalars are denoted

$$R \equiv R_{MN}G^{MN}\,, \qquad \mathcal{R} \equiv \mathcal{R}_{\mu\nu}g^{\mu\nu}\,, \qquad \tilde{\mathcal{R}} \equiv \tilde{\mathcal{R}}_{mn}g^{mn}\,. \tag{2.4}$$

Taking partial traces in eqs. (2.2) and (2.3) yields

$$\frac{1}{p+1}\left(\mathcal{R}W^{-2} - R^{\mu}{}_{\mu}\right) = pW^{-2}\nabla W\cdot\nabla W + W^{-1}\nabla^2 W\,, \tag{2.5}$$

$$\frac{1}{p+1}\left(\tilde{\mathcal{R}} - R^{m}{}_{m}\right) = W^{-1}\nabla^2 W\,, \tag{2.6}$$

where we use the notation: $R^{\mu}{}_{\mu} \equiv W^{-2}g^{\mu\nu}R_{\mu\nu}$ and $R^{m}{}_{m} \equiv g^{mn}R_{mn}$. Therefore $R = R^{\mu}{}_{\mu} + R^{m}{}_{m}$. Now, consider the following total derivative on the internal space:

$$\nabla\cdot(W^{\alpha}\nabla W) = W^{\alpha+1}\left[\alpha W^{-2}\nabla W\cdot\nabla W + W^{-1}\nabla^2 W\right], \tag{2.7}$$

where α is an arbitrary constant. Comparing this with the r.h.s.'s of eqs. (2.5) and (2.6), we find that we can write the total derivative as

$$\nabla\cdot(W^{\alpha}\nabla W) = \frac{W^{\alpha+1}}{(p+1)p}\left[\alpha(\mathcal{R}W^{-2} - R^{\mu}{}_{\mu}) + (p-\alpha)(\tilde{\mathcal{R}} - R^{m}{}_{m})\right]. \tag{2.8}$$

The full D-dimensional Einstein equations may be written as

$$R_{MN} = 8\pi G_D\left(T_{MN} - \frac{1}{D-2}G_{MN}T^{P}{}_{P}\right), \tag{2.9}$$

where G_D is the D-dimensional gravitational constant. Using these, we can write $R^\mu{}_\mu$ and $R^m{}_m$ in terms of the stress-energy tensor:

$$R^\mu{}_\mu = \frac{8\pi G_D}{D-2}\left((D-p-3)T^\mu{}_\mu - (p+1)T^m{}_m\right), \tag{2.10}$$

$$R^m{}_m = \frac{8\pi G_D}{D-2}\left((p-1)T^m{}_m - (D-p-1)T^\mu{}_\mu\right), \tag{2.11}$$

where we have used $T^M{}_M = T^\mu{}_\mu + T^m{}_m$ — that is, $T^\mu{}_\mu \equiv W^{-2}g^{\mu\nu}T_{\mu\nu}$, in analogy with the above. Substituting eqs. (2.10) and (2.11) into eq. (2.8), the total derivative becomes

$$\nabla\cdot(W^\alpha\nabla W) = \frac{W^{\alpha+1}}{(p+1)p}\left\{\frac{8\pi G_D}{D-2}\Big(T^\mu{}_\mu[(p-2\alpha)(D-p-1)+2\alpha]+ \right. \tag{2.12}$$
$$\left. + T^m{}_m p(2\alpha-p+1)\Big) + (p-\alpha)\tilde{\mathcal{R}} + \alpha\mathcal{R}W^{-2}\right\}.$$

If we have a compact internal space, the integral of the l.h.s. vanishes.[2] We are then left with

$$\oint W^{\alpha+1}\Big(T^\mu{}_\mu[(p-2\alpha)(D-p-1)+2\alpha] + T^m{}_m\, p(2\alpha-p+1) +$$
$$+\frac{D-2}{8\pi G_D}[(p-\alpha)\,\tilde{\mathcal{R}} + \alpha\,\mathcal{R}\,W^{-2}]\Big) = 0\,, \tag{2.13}$$

which is a constraint that must be satisfied by any consistent brane world model. Setting $\alpha = n-1$, $p=3$ and $D=5$ (for which $\tilde{\mathcal{R}}=0$) reproduces the consistency conditions derived in ref. [5]. Eq. (2.13) provides a generalization of their work, which in particular is not limited to internal spaces of one dimension. Finally, if the internal space is not compact, eq. (2.13) may still provide an interesting consistency condition, as long as care is taken with the boundary conditions.

We wish to apply this condition to various brane world scenarios to test their consistency with Einstein's equations. With this in mind, we write an ansatz for the stress-energy tensor of the form

$$T_{MN} = -\frac{\Lambda G_{MN}}{8\pi G_D} - \sum_i T_q^{(i)} P[G_{MN}]_q^{(i)}\Delta^{(D-q-1)}(y-y_i) + \mathcal{T}_{MN}\,. \tag{2.14}$$

As well as a bulk cosmological constant, this describes a collection of branes of various dimensions. The i^{th} brane is a q-brane (with $q \geq p$) with tension $T_q^{(i)}$ (with units energy/length^q) and transverse coordinates y_i. $P[G_{MN}]_q^{(i)}$ is the pull-back of the spacetime metric to the worldvolume of the q-brane. Any other bulk or worldvolume matter field contributions are implicitly encoded in $\mathcal{T}_{MN}$. In this ansatz, $\Delta^{(D-q-1)}(y-y_i)$ denotes that covariant combination of delta functions and (geo)metric factors

[2] Some attention should be paid to the smoothness of the warp factor.

necessary to position the brane. Typically, this will be a product of terms of the form $\delta(y-y_i)/\sqrt{G_{yy}}$, but a more sophisticated expression may be required if some of the relevant coordinates are ignorable at the position of the brane — see appendix A. Note that we are implicitly assuming that all of the branes are extended in all of the x^μ directions, and, if $q>p$ for a particular brane, it spans a $(q-p)$-cycle in the internal space.

Given this ansatz, we deduce that

$$T^\mu{}_\mu = -(p+1)\left[\frac{\Lambda}{8\pi G_D} + \sum_i T_q^{(i)}\Delta^{(D-q-1)}(y-y_i)\right] + \mathcal{T}^\mu{}_\mu\,, \tag{2.15}$$

$$T^m{}_m = -(D-p-1)\frac{\Lambda}{8\pi G_D} - \sum_i (q-p)\,T_q^{(i)}\Delta^{(D-q-1)}(y-y_i) + \mathcal{T}^m{}_m\,. \tag{2.16}$$

The consistency condition (2.13) may now be written as

$$\begin{aligned}\oint W^{\alpha+1}\Big(\alpha\,\mathcal{R}\,W^{-2} + (p-\alpha)\,\tilde{\mathcal{R}} - [\gamma + (D-p-1)\,\tilde{\gamma}]\,\Lambda - \qquad\qquad (2.17)\\ -8\pi G_D\left[\sum_i\left(\gamma+(q-p)\tilde{\gamma}\right)T_q^{(i)}\Delta^{(D-q-1)}(y-y_i) - \frac{\gamma}{p+1}\mathcal{T}^\mu{}_\mu - \tilde{\gamma}\mathcal{T}^m{}_m\right]\Big) = 0\,,\end{aligned}$$

where we have introduced the following constants:

$$\gamma = \frac{p+1}{D-2}\left[(p-2\alpha)(D-p-1)+2\alpha\right]\,, \qquad \tilde{\gamma} = \frac{p(2\alpha-p+1)}{D-2}\,. \tag{2.18}$$

Dimensional parameters aside, eq. (2.17) gives a one parameter (α) family of consistency conditions relating the geometry of the brane world to its stress-energy content. As such, this is merely a convenient re-expression of certain components of Einstein's equations, with a stress-energy tensor of the form (2.14). These general results are not particularly transparent and so to get a better insight into what these sum rules are telling us, we will now specialize to the phenomenologically interesting case $p=3$. Focusing on the choice $\alpha=-1$ also simplifies the expressions because the warp factor is removed from all of the terms except that involving $\mathcal{R}$, the Ricci scalar for the noncompact metric $g_{\mu\nu}$. With these choices, eq. (2.17) reduces to

$$\begin{aligned}\oint\Big(-\mathcal{R}W^{-2} + 4\tilde{\mathcal{R}} - \frac{8(D-5)}{D-2}\Lambda + \frac{8\pi G_D}{D-2}\left[(5D-22)\mathcal{T}^\mu{}_\mu - 12\mathcal{T}^m{}_m\right]\Big) =\\ = \frac{32\pi G_D}{D-2}\sum_i (5D-13-3q)L_i\,T_q^{(i)}\,,\end{aligned} \tag{2.19}$$

where L_i is the area of the $(q-p)$-cycle in the internal space spanned by the i^{th} brane. If $q=p$ (i.e., the brane is not extended in the internal space), then $L_i=1$.

Let us first consider the case $D = 5$. The above constraint then simplifies, since the coefficient of the Λ contribution vanishes and $\tilde{\mathcal{R}} = 0$ since there is a single internal direction. If we set aside the additional contributions of matter fields (i.e., set $\mathcal{T}^\mu{}_\mu = 0 = \mathcal{T}^m{}_m$), the constraint becomes

$$-\mathcal{R} \oint W^{-2} = 32\pi G_5 \sum_i T_3^{(i)} . \tag{2.20}$$

Our result here essentially reproduces that given in ref. [5] — compare to their eq. (2.26). In particular, if the curvature on the branes is positive or vanishes, we have $\sum_i T_3^{(i)} \leq 0$ and so we must include some number of negative tension branes for a consistent model.

Note that in general $\mathcal{R} = \mathcal{R}(x)$ is independent of the internal coordinates y^n, hence it appears outside of the integration in eq. (2.20). However, with the restrictions imposed above (i.e., $\mathcal{T}_{MN} = 0$), it must further be true that in fact $\mathcal{R}$ is a fixed constant. The same will be true in all of the examples considered in the following.

However, let us consider the constraint with $D = 6$ but again no matter fields for simplicity. With these choices, eq. (2.19) becomes

$$\oint \left(-\mathcal{R} W^{-2} + 4\tilde{\mathcal{R}} - 2\Lambda \right) = 8\pi G_6 \sum_i (17 - 3q) L_i T_q^{(i)} . \tag{2.21}$$

Note that on the r.h.s., there are contributions coming from three- and four-branes, both with positive coefficients. On the l.h.s., however, we also have contributions coming from the cosmological constant and the curvature of the two-dimensional internal space. Certainly, these contributions afford us much more leeway in constructing consistent brane world models, even when no matter fields are present. For instance, a positive $\tilde{\mathcal{R}}$ and negative Λ can produce an overall positive contribution on the l.h.s. which could then accommodate the appearance of only positive tensions on the r.h.s. Similar contributions from the cosmological constant and internal curvature appear in eq. (2.19) for all higher dimensions $D \geq 6$. Hence, the sum rules are obviously much less restrictive when we go beyond $D = 5$. We shall explore a few examples in the following section.

3. $D = 6$ brane world examples

As an application of the sum rules (2.17) derived above, we consider two examples with $D = 6$ and $p = 3$. Other interesting examples may be found in refs. [10, 14, 15, 16]. In particular, refs. [10, 14] provide compactifications involving only positive tension branes. With the present choice of dimensions, the constants in eq. (2.18) become

$$\gamma = 2(3 - \alpha) , \qquad \tilde{\gamma} = \frac{3}{2}(\alpha - 1) , \qquad \gamma + (D - p - 1)\, \tilde{\gamma} = 3 + \alpha . \tag{3.1}$$

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For matter fields, we only consider a gauge field whose field strength is proportional to the volume form of the two-dimensional internal space, i.e., $F_{mn} = k\epsilon_{mn}$. The traces of the corresponding stress-energy tensor are then

$$\mathcal{T}^{\mu}{}_{\mu} = -k^2\,, \qquad \mathcal{T}^{m}{}_{m} = \frac{k^2}{2}\,. \tag{3.2}$$

The general consistency conditions (2.17) then become

$$\begin{aligned}\oint W^{\alpha+1}\left(\alpha\,\mathcal{R}\,W^{-2} + (3-\alpha)\,\tilde{\mathcal{R}} - (3+\alpha)\Lambda - 2\pi G_6(9-5\alpha)k^2\right) &= \\ = 4\pi G_6\left(4(3-\alpha)\sum_i T_3^{(i)} W_i^{\alpha+1} + (9-\alpha)\sum_i T_4^{(i)}\oint_i W^{\alpha+1}\right)\end{aligned} \tag{3.3}$$

where $W_i = W(y = y_i)$ and $\oint_i W^{\alpha+1}$ denotes an integral over the one-cycle spanned by the i^{th} four-brane in the internal space.

Considering again the choice $\alpha = -1$, we have a slight generalization of eq. (2.21):

$$\alpha = -1: \; -\mathcal{R}\oint W^{-2} + 4\oint \tilde{\mathcal{R}} - 2\Lambda V_2 - 28\pi G_6 k^2 V_2 = 8\pi G_6\left(8\sum_i T_3^{(i)} + 5\sum_i L_i T_4^{(i)}\right). \tag{3.4}$$

Here we have introduced V_2 to denote the volume of the internal space, which appears in the contributions from the cosmological constant and the gauge field. In this particular equation, the integral over the internal curvature deserves special attention because it yields a topological invariant, the Euler characteristic:

$$\chi = \frac{1}{4\pi}\oint \tilde{\mathcal{R}}\,. \tag{3.5}$$

Hence, this contribution yields a simple global constant which characterizes the six-dimensional model of interest.

Returning to the general expression (3.3), we see that certain choices of α will cause one or more of the contributions to vanish. We consider explicitly the following cases:

$$\begin{aligned}\alpha = -3: \quad & \oint W^{-2}\left(-\mathcal{R}W^{-2} + 2\tilde{\mathcal{R}} - 16\pi G_6 k^2\right) = \\ & = 16\pi G_6\left(2\sum_i T_3^{(i)} W_i^{-2} + \sum_i T_4^{(i)}\oint_i W^{-2}\right), \end{aligned} \tag{3.6}$$

$$\alpha = 0: \quad \oint W\left(\tilde{\mathcal{R}} - \Lambda - 6\pi G_6 k^2\right) = 4\pi G_6\left(4\sum_i T_3^{(i)} W_i + 3\sum_i T_4^{(i)}\oint_i W\right), \tag{3.7}$$

$$\begin{aligned}\alpha = \frac{9}{5}: \quad & \oint W^{14/5}\left(3\mathcal{R}W^{-2} + 2\tilde{\mathcal{R}} - 8\Lambda\right) = \\ & = 16\pi G_6\left(2\sum_i T_3^{(i)} W_i^{14/5} + 3\sum_i T_4^{(i)}\oint_i W^{14/5}\right), \end{aligned} \tag{3.8}$$

$$\alpha = 3: \quad \oint W^4 \left(\mathcal{R} W^{-2} - 2\Lambda + 4\pi G_6 k^2\right) = 8\pi G_6 \sum_i T_4^{(i)} \oint_i W^4 \,, \tag{3.9}$$

$$\alpha = 9: \quad \oint W^{10} \left(3\mathcal{R} W^{-2} - 2\tilde{\mathcal{R}} - 4\Lambda + 24\pi G_6 k^2\right) = -32\pi G_6 \sum_i T_3^{(i)} W_i^{10} \,. \tag{3.10}$$

We will comment on these expressions for the following two examples.

3.1 Non-warped example

We consider first the example of non-warped compactifications, i.e., compactifications with a factorizable spacetime manifold, as arose in the original discussion of large extra dimensions [1]. That is, we set $W = 1$ everywhere. Of course, this greatly simplifies the consistency conditions above.

One of the interesting aspects of six-dimensional models is that three-branes are co-dimension two objects. Therefore, the effect of a relativistic three-brane on the spacetime geometry is to induce an angular deficit in the transverse space, in analogy to the effect of a cosmic string in four dimensions [17]. That is, locally the geometry of the internal space is a cone with the three-brane located at the tip [18]. Einstein's equations relate the local deficit angle to the tension of the three-brane by

$$\delta_i = 8\pi G_6 T_3^{(i)} \,. \tag{3.11}$$

As a result of this simple geometric effect, it is straightforward to produce consistent compactifications which involve only flat three-branes. If we include only three-branes and set $\Lambda = 0 = k = \mathcal{R}$ (as well as $W = 1$), then all of the above consistency conditions yield a single nontrivial constraint,

$$\chi = 4 G_6 \sum_i T_3^{(i)} \,, \tag{3.12}$$

where we have used eq. (3.5). If the internal space has a spherical topology, so that $\chi = 2$, we can construct a brane world model with only positive tension three-branes. Upon using eq. (3.11), this consistency condition becomes

$$\sum_i \delta_i = 4\pi \,, \tag{3.13}$$

which was previously noted in this context in ref. [19]. If instead we consider a torus ($\chi = 0$), as is usually considered in these scenarios [1], consistency would demand that we introduce a number of negative tension three-branes. The same would be true for internal spaces of higher genus, as arise, e.g., in the compactifications considered in ref. [20]. Note that in all the cases discussed here, the spacetime is locally completely flat. The three-branes only introduce delta function curvature distributions in the transverse space which produce a compact internal manifold.

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Note that eq. (3.12) puts no constraints on the number of branes that might appear in, e.g., the two-sphere compactification. Geometrically, however, there is a lower bound. If we want a smooth internal space with a finite volume, we would need at least three branes or curvature sources at distinct positions. Of course, this is essential if the model is to produce interesting phenomenology, as the four-dimensional Newton's constant G_4 is related to that in six dimensions by [1]

$$G_4 = \frac{G_6}{V_2}\,. \tag{3.14}$$

Let us now generalize the discussion of these factorizable scenarios by including the contributions of the cosmological constant and the gauge field flux on the internal space. We still only include flat three-branes with $\mathcal{R} = 0$. In this case, the previous analysis imposes two nontrivial constraints. The first comes most directly from eq. (3.9) which yields

$$\Lambda = 2\pi G_6 k^2 > 0\,. \tag{3.15}$$

Essentially, this constraint on the internal flux and Λ ensures that Einstein's equations are still satisfied everywhere with a Ricci-flat brane metric $g_{\mu\nu}$. The remaining constraint generalizes eq. (3.12) to

$$\chi - 2G_6 V_2 k^2 = 4G_6 \sum_i T_3^{(i)}\,. \tag{3.16}$$

Hence, in some sense, the introduction of these extra parameters makes harder the construction of consistent compactifications with only positive tension branes. Of course, consistent models are still possible with a compactification on a two-sphere. Ref. [21] provides an explicit realization as follows: After a compactification on a round two-sphere where eq. (3.15) is satisfied, we cut out a wedge along two meridians. Pasting these two edges introduces angular deficits representing three-branes with equal tensions, at the north and south poles. Of course, more elaborate constructions involving more three-branes would also be possible.

3.2 Brane worlds from the AdS soliton

In this section, we consider a less trivial six-dimensional configuration in order to demonstrate the use of the consistency conditions. We begin by describing, in some detail, the geometry of the AdS soliton [9]. This is an asymptotically locally AdS solution of Einstein's equations with a negative cosmological constant. We demonstrate how, by applying a cutting-and-pasting procedure analogous to that used in the Randall-Sundrum scenario [3, 4] (see discussion in ref. [22]), the AdS soliton yields a brane world model with a two-dimensional internal space of spherical topology. A similar model was considered earlier in ref. [10] from a different point of view. Finally we consider below the consistency conditions as they apply to the resulting solution.

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The AdS soliton can be constructed by doubly analytically continuing the metric for a planar AdS-Schwarzschild black hole in D dimensions. Here we focus our attention on the case $D = 6$ which will lend itself to producing a (3+1)-dimensional brane world. In horospheric coordinates, the line element is

$$ds^2 = \frac{r^2}{L^2}(\eta_{\mu\nu}dx^\mu dx^\nu + f^2(r)d\tau^2) + \frac{L^2}{r^2}\frac{dr^2}{f^2(r)}\,, \tag{3.17}$$

where $f^2(r) = 1 - \omega^5/r^5$ and $\eta_{\mu\nu}$ is the four-dimensional Minkowski metric. L is related to the cosmological constant by $\Lambda = -10/L^2$. By comparison with eq. (2.1), the warp factor is $W(r) = r/L$. The τ coordinate, which will become a part of an internal space, will be periodically identified with period

$$\ell = \frac{2L^2}{5\omega}\,(2\pi - \delta)\,. \tag{3.18}$$

As $f(r)$ vanishes at $r = \omega$, this circle shrinks to zero size and the geometry closes off there. Hence, in the basic AdS soliton, we only consider the radial coordinate for the range $r \geq \omega$. The parametrization was chosen in eq. (3.18) such that if $\delta = 0$, the geometry is actually smooth at $r = \omega$. Otherwise, the spacetime exhibits a conical singularity in the (r, τ) space, with an angular deficit δ at $r = w$. In the current context, we interpret this geometry as arising from a three-brane positioned at the tip of the cone with tension

$$\delta = 8\pi G_6 T_3\,, \tag{3.19}$$

in analogy to the discussion in the previous subsection.

To construct a brane world geometry with a compact internal space, we can truncate the AdS soliton at some value of the radius, say $r = R$, and paste two copies of the inner region along the cutting surface. The r and τ coordinates then form an internal space with spherical topology. In the following, we actually consider a slightly more elaborate configuration by letting the two regions which are being pasted together have different parameters, ω, R and δ.[3] To paste together the two geometries smoothly requires us to match the metrics induced on the interface by the different embeddings. Consider joining one copy with $\omega = \omega_1$, $R = R_1$ and $\delta = \delta_1$ to another with $\omega = \omega_2$, $R = R_2$ and $\delta = \delta_2$. At the interface the x^μ coordinates in the second copy can be Lorentz transformed so that they are aligned with those in the first copy, but in general the warp factors in the two geometries will differ at the interface. Hence, matching the metrics will require that we introduce a scaling $x_2^\mu = \beta x_1^\mu$ where

$$\beta = \frac{W(r_1 = R_1)}{W(r_2 = R_2)} = \frac{R_1}{R_2}\,. \tag{3.20}$$

[3] The model of ref. [10] corresponds to the case where a Z_2 symmetry is imposed at the boundary. A further extension of our construction would be to allow L or the cosmological constant have different values on either side of the interface, as might happen for certain types of domain walls.

Similarly, the τ coordinates on either side should be scaled to match across the interface: $\tau_2 = \ell_2\tau_1/\ell_1$. Further, the metric parameters must be constrained in order that the proper period of the circle direction is the same in both of the geometries:

$$\left(\frac{R_1 L_1(2\pi-\delta_1)}{\omega_1}\right)^2\left(1-\frac{\omega_1^5}{R_1^5}\right)=\left(\frac{R_2 L_2(2\pi-\delta_2)}{\omega_2}\right)^2\left(1-\frac{\omega_2^5}{R_2^5}\right). \tag{3.21}$$

Imposing these conditions will ensure continuity of the metric at the interface, but in general, it will not be differentiable. The discontinuity in the extrinsic curvature across the interface is interpreted as resulting from a delta function source of stress-energy distributed over this hypersurface [23] — see also ref. [24]. The surface stress-energy tensor S_{AB} may be calculated as

$$8\pi G_6 S_{AB} = (K^{(2)}+K^{(1)})_{AB} - G_{AB}\,(K^{(2)}+K^{(1)})^C{}_C\,, \tag{3.22}$$

where $K^{(i)}_{AB} = n_i\partial_{r_i}G_{AB}/2$ are the extrinsic curvatures of the interface due to its embedding in each of the geometries. Note that the latter formula for the extrinsic curvature applies because of the diagonal form of the metric. The normalization factors in this expression are given by

$$n_i = \frac{R_i}{L}\sqrt{1-\frac{\omega^5}{R_1^5}}\,. \tag{3.23}$$

The surface stress-energy is then found to have components

$$S_{\mu\nu} = -\frac{\eta_{\mu\nu}}{16\pi G_6 L^3}R_1^2\left\{\left(1-\frac{\omega_1^5}{R_1^5}\right)^{-1/2}\left[8-3\frac{\omega_1^5}{R_1^5}\right]+\left(1-\frac{\omega_2^5}{R_2^5}\right)^{-1/2}\left[8-3\frac{w_2^5}{R_2^5}\right]\right\} \tag{3.24}$$

$$S_{\tau\tau} = -\frac{1}{2\pi G_6 L^3}\left\{\left(1-\frac{w_1^5}{R_1^5}\right)^{3/2}R_1^2-\left(1-\frac{\omega_2^5}{R_2^5}\right)^{3/2}\left[R_2\frac{\ell_2}{\ell_1}\right]^2\right\}, \tag{3.25}$$

where use has been made of the matching conditions (3.20) and (3.21).

To interpret this surface stress-energy as arising from a(n infinitely thin) relativistic four-brane, it should have the form $S_{AB} = -T_4 G_{AB}$. Unfortunately, eqs. (3.24) and (3.25) do not accommodate this simple interpretation. However, the problem is solved naturally by assuming that the source is composed of a bound state of a four-brane and a three-brane de-localised around the circle direction. If we assume that the brane tensions combine linearly,[4] the stress-energy would have the form:

$$S_{\mu\nu} = -\left(T_4+\frac{T_3}{L_\tau}\right)G_{\mu\nu}\Big|_{\text{interface}}\,, \tag{3.26}$$

$$S_{\tau\tau} = -T_4\,G_{\tau\tau}|_{\text{interface}}\,, \tag{3.27}$$

[4]This would result if the four-brane theory involved a three-form gauge potential with a simple quadratic action: $\mathcal{L}_4 = -T_4\sqrt{G}(1+F^2)$.

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where L_τ is the proper period of the circle direction, i.e., $L_\tau = \ell_1 f(R_1)R_1/L$. We are assuming that the three-brane is extended in the x^μ directions, but is also de-localised around the τ circle. With this ansatz, we find the tensions to be

$$T_4 = \frac{1}{2\pi G_6 L}\left\{\left(1-\frac{\omega_1^5}{R_1^5}\right)^{1/2} + \left(1-\frac{\omega_2^5}{R_2^5}\right)^{1/2}\right\}, \tag{3.28}$$

$$T_3^{(3)} = \frac{1}{8\pi G_6}\left\{\left(\frac{\omega_1}{R_1}\right)^4 (2\pi-\delta_1) + \left(\frac{\omega_2}{R_2}\right)^4 (2\pi-\delta_2)\right\}. \tag{3.29}$$

Although T_4 displays no explicit dependence upon the deficit angles, such a dependence is implicit through the matching condition (3.21). Note that both these tensions are positive. We have added a superscript 3 to the three-brane tension, as we have already introduced three-branes at the two conical singularities at $r_i = \omega_i$ with tensions proportional to their deficit angles: $T_3^{(i)} = \delta_i/(8\pi G_6)$ with $i = 1, 2$. In this way, we see that the AdS soliton provides a basic building block for constructing highly tunable brane world spacetimes where, in principle, any of the three three-branes might be considered as the "visible" one.

To further study the details of our six-dimensional brane world, it is convenient to define a new radial coordinate

$$\rho = \begin{cases} r_1 & \text{in the first copy of the AdS soliton,} \\ R_1 + R_2 - r_2 & \text{in the second copy of the AdS soliton.} \end{cases} \tag{3.30}$$

Then the τ circle closes off at $\rho = \omega_1$, the interface lies at $\rho = R_1$, and the second copy closes off at $\rho = R_1 + R_2 - \omega_2$. In terms of this coordinate, the warp factor is

$$W(\rho) = \frac{\rho}{L}\theta(R_1-\rho) + (R_1+R_2-\rho)\frac{\beta}{L}\theta(\rho-R_1). \tag{3.31}$$

The stress-energy tensor as given in eq. (2.14) becomes

$$T_{MN} = -\frac{\Lambda G_{MN}}{8\pi G_6} - \sum_i^3 T_3^{(i)} P[G_{MN}]_3^{(i)} \Delta^{(2)}(y-y_i) - T_4 P[G_{MN}]_4 \Delta^{(1)}(y-y_4), \tag{3.32}$$

where the Δ-functions are given by (see appendix A)

$$\begin{aligned} \Delta^{(2)}(y-y_1) &= \frac{\delta(\rho-\omega_1)}{\ell_1}, & \Delta^{(2)}(y-y_2) &= \frac{\delta(\rho-[R_1+R_2-\omega_2])}{\ell_2}, \\ \Delta^{(2)}(y-y_3) &= \frac{\delta(\rho-R_1)}{L_\tau\sqrt{G_{\rho\rho}}}, & \Delta^{(1)}(y-y_4) &= \frac{\delta(\rho-R_1)}{\sqrt{G_{\rho\rho}}}. \end{aligned} \tag{3.33}$$

Finally, the internal volume is given by $V_2 = \ell_1(R_1-\omega_1) + \ell_2(R_2-\omega_2)$, and the internal curvature can be shown to be

$$\begin{aligned} \tilde{\mathcal{R}} = {} & 16\pi G_6 \sum_i^3 T_3^{(i)} \Delta^{(2)}(y-y_i) + 4\pi G_6 T_4 \Delta^{(1)}(y-y_4) - \\ & - 2\left(\frac{\rho^5 - 6\omega_1^5}{\rho^5 L^2}\right)\theta(R_1-\rho) - 2\left(\frac{(R_1+R_2-\rho)^5 - 6\omega_2^5}{(R_1+R_2-\rho)^5 L^2}\right)\theta(\rho-R_1), \end{aligned} \tag{3.34}$$

either by using eq. (2.6) and subsequent formulae from section two, or from the intrinsic properties of the internal space.

We are now in a position to verify the sum rules (3.4) and (3.6)–(3.10). The simplest of these is eq. (3.4), which becomes

$$16\pi + V_2|\Lambda| = 32\pi G_6 \sum_{i=1}^{3} T_3^{(i)} + 20\pi G_6 L_\tau T_4 \,. \tag{3.35}$$

Given the results above, an explicit calculation verifies that this constraint, as well as the remaining consistency conditions are, indeed, satisfied. Of course, this is actually a consistency check of our different calculations as we began this subsection by showing how the brane world based on the AdS soliton satisfies Einstein's equations.

4. Discussion

We have extended the brane world consistency conditions derived in ref. [5] for five dimensions to theories of an arbitrary spacetime dimension. Ultimately, these sum rules amount to a clever re-expression of certain components of Einstein's equations. However, they prove very powerful in characterizing five-dimensional models, as it was shown quite generally that a consistent compactification should include negative tension branes. The sum rules become less restrictive for $D \geq 6$, as illustrated in eq. (2.19). The essential new ingredient in higher dimensions is that the internal space may have nontrivial curvature which contributes on the l.h.s. of Einstein's equations. Hence, we found that a model with a positively curved internal space (i.e., $\tilde{\mathcal{R}} > 0$) can be consistently constructed with only positive tension branes. In section 3.1, the non-warped compactifications on a two-sphere provide examples where the internal curvature precisely matches the contribution from the positive tension branes in the consistency conditions.

In eq. (2.19), we see that the coefficient of the cosmological constant happens to vanish in precisely five dimensions but is negative for $D \geq 6$. Hence in higher dimensions, a negative cosmological constant is another ingredient which helps in producing consistent compactifications with only positive tension branes. This played a role in the warped compactifications of section 3.2, which were based on the AdS soliton.

An interesting feature of the six-dimensional models is that three-branes are co-dimension two objects and so only introduce isolated curvature defects in the internal space. There is a very simple relation between the three-brane tension and the internal geometry, as illustrated in eq. (3.11). For higher dimensions (i.e., $D > 6$), the curvature generated by the self-gravity of the three-branes is no longer localized. In particular, if a three-brane was allowed to become arbitrarily thin, it would be surrounded by an event horizon with $r_H^{D-6} \sim G_D T_3$. Therefore a realistic brane world must introduce a model in which the three-branes have a thickness larger than this horizon radius, and the curvature in the internal space becomes dependent on this model for the internal structure of the branes — see, for example, the discussions in ref. [25]. Hence, the sum rules lose much of their power, in that it is much harder to derive statements that cover a broad class of models. Of course, we can always consider the sum rule (2.17) with $\alpha = p$, in which the coefficient of the internal curvature vanishes. With this choice, the total derivative in eq. (2.7) is precisely that appearing already in the the components of the Ricci tensor with brane coordinate indices (i.e., $R_{\mu\nu}$), as given in eq. (2.2). For the phenomenologically interesting case of $p = 3$, the corresponding sum rule is

$$\mathcal{R}\oint W^2 - \frac{8}{D-2}\Lambda\oint W^4 = \frac{32\pi G_D}{D-2}\left[\sum_i (6-D)T_3^{(i)}W_i^4 + \sum_{i,q>3}(q+3-D)T_q^{(i)}\oint_i W^4\right], \tag{4.1}$$

where we have not included any matter field contributions. For $D > 6$, this equation tells us that if we wanted to construct a consistent compactification with only flat (i.e., $\mathcal{R} = 0$) three-branes, we would have to include a *positive* cosmological constant in the theory. Of course, this equation does not guarantee that such a solution exists but only provides a necessary condition for consistency. For example, if we applied the same reasoning to $D = 5$, we would conclude that a negative cosmological constant is necessary. However, examining other consistency conditions, e.g., eq. (2.20), tells us that a consistent solution with only flat, positive tension three-branes is impossible in five dimensions, independent of the sign or magnitude of the cosmological constant. Note that for $D = 6$ such a compactification would not be possible unless $\Lambda = 0$, which was the case for our non-warped examples. For the warped $D = 6$ example based on the AdS soliton, we have a negative cosmological constant but its contribution in eq. (4.1) is balanced by that of the central four-brane on the r.h.s.

Given that the results of the sum rules are less restrictive in higher dimensions, one might also gain insight by establishing inequalities as follows: Multiply the total derivative in eq. (2.7) by $W^{-\beta}$ and then integrate over the internal manifold. After integrating by parts, one finds

$$\beta\oint W^{\alpha-\beta-1}(\nabla W)^2 \geq 0\,, \tag{4.2}$$

where the inequality assumes that β is positive, and it will only be saturated if W is a constant. Hence following the same analysis as in section 2, eq. (2.17) becomes an inequality with the l.h.s. being greater than or equal to zero. The only modification to the integrand is that the initial factor of $W^{\alpha+1}$ is replaced by $W^{\alpha+1-\beta}$. Hence we have the freedom to eliminate this term by choosing $\beta = \alpha + 1$ producing an expression where the warp factor only appears in the first term involving the brane curvature. We can again eliminate the contribution from the internal curvature by choosing $\alpha = p$, as above. In this case, with $p = 3$ (and $\mathcal{T}_{MN} = 0$), eq. (4.1) is replaced by

$$\mathcal{R} \oint W^{-2} - \frac{8}{D-2}\Lambda V_{D-4} \geq \frac{32\pi G_D}{D-2}\left[\sum_i (6-D)T_3^{(i)} + \sum_{i,q>3}(q+3-D)T_q^{(i)} L_i\right]. \tag{4.3}$$

While less precise than the sum rules, inequalities such as these were sufficient to establish certain no-go theorems [11, 26] and also played a role in guiding the construction of ref. [27].

The warped brane world based on the AdS soliton deserves further comment as it may be useful in providing a phenomenologically interesting scenario. First we remark that there are a number of straightforward extensions of the construction described in section 3.2. First of all, AdS soliton solutions exist for arbitrary dimensions, and so this construction can be generalized to produce a warped compactification for arbitrary p. Of course, some of the additional x^μ would then have to be compactified to produce a four-dimensional effective theory at low energies. The construction can also be extended to include a magnetic flux on the compact space by beginning with the analogous AdS soliton constructed by an appropriate analytical continuation of the AdS-Reissner-Nordström black hole. Ref. [10] also considered the extension of these compactifications such that the brane world has a cosmological metric, similar to the discussions of ref. [28]. From our point of view, the essential step is to begin with the standard AdS-Schwarzschild black hole where the horizon topology is $S^{D-2} \times R$, rather than R^{D-1} as for the planar black hole. Analytically continuing and then performing the cut-and-paste construction results in a brane world where the geometry of the three-branes corresponds to de Sitter space. Alternatively, anti-de Sitter branes can be produced if one begins with a "topological" black hole where the horizon has negative intrinsic curvature [29]. We should also mention that a portion of the AdS soliton geometry appeared in a more elaborate cut-and-paste construction in ref. [15].

One comment about our warped model is that the low energy theory will include precisely four-dimensional Einstein gravity. This observation comes from the fact that the initial AdS soliton metric (3.17) can be generalized to

$$ds^2 = \frac{r^2}{L^2}\left(g_{\mu\nu}dx^\mu dx^\nu + f^2(r)d\tau^2\right) + \frac{L^2}{r^2}\frac{dr^2}{f^2(r)}. \tag{4.4}$$

This metric, with the same function $f^2(r) = 1 - \omega^5/r^5$, still satisfies the six-dimensional Einstein's equations with a negative cosmological constant, as long as the brane metric $g_{\mu\nu}$ is Ricci flat (i.e., $\mathcal{R}_{\mu\nu}(g) = 0$). That is, the brane metric must satisfy the (fully nonlinear) vacuum Einstein equations in four dimensions. Curving the brane metric in this way to generalize the original solution is a relatively general property that applies to warped compactifications which display four-dimensional

Poincaré invariance [30]. Therefore the warping does not disturb the emergence of the standard Einstein gravity in the low energy theory. However, the warp factor does modify the naive relation between gravitational coupling in four and six dimensions:

$$G_6 = G_4 \left(\frac{R_1^3 - \omega_1^3}{3L^2} \ell_1 + \frac{R_2^3 - \omega_2^3}{3L^2} \ell_2 \right) . \tag{4.5}$$

In comparing this result to eq. (3.14), recall that $V_2 = (R_1 - \omega_1)\ell_1 + (R_2 - \omega_2)\ell_2$ for our warped compactification.

From the point of view of linearized fluctuations, the above discussion indicates that four-dimensional gravitons in the brane metric will be a zero mode of our warped brane world. Using a linearized analysis similar to that of ref. [31], one finds that there are no other zero mode fluctuations in the internal metric [32]. That is, there are no "scalar" zero modes that correspond to varying the size or geometry of the internal space. Hence once the cosmological constant and the brane tensions are fixed, there is a unique solution for the internal space. This is interesting because by going beyond five dimensions, not only have we eliminated the need for negative tension branes, we have also managed to stabilize the internal space! This is not a generic feature of higher dimensional compactifications. There are many moduli in the non-warped example corresponding to both the volume of the compact space and the relative position of the three-branes. It would be interesting to better understand what features of the AdS soliton model were essential in stabilizing the internal space.

An interesting lesson of the RS I scenario [3] is that one can produce a large hierarchy from the gravitational redshift between branes, as might arise in a warped compactification. The warped compactification considered here provides another realization of this effect [10] (without the need to introduce negative tension branes). To make this feature manifest, we perform the following coordinate transformation,

$$y(r) = \frac{2L}{5} \text{arccosh} \left(\frac{r}{\omega} \right)^{5/2} , \tag{4.6}$$

on either side of the interface, which puts the metric (3.17) in the form

$$ds^2 = W^2(y) \left(\eta_{\mu\nu} dx^\mu dx^\nu + \tanh^2 \left(\frac{5y}{2L} \right) d\tau^2 \right) + dy^2 \,, \tag{4.7}$$

where the warp factor is now

$$W(y) = \frac{\omega}{L} \cosh^{2/5} \left(\frac{5y}{2L} \right) . \tag{4.8}$$

In this new coordinate system, the range $\omega \leq r \leq R$, where R is the position of the bound three/four-brane system, corresponds to

$$0 \leq y \leq \frac{2L}{5} \text{arccosh} \left(\frac{R}{\omega} \right)^{5/2} . \tag{4.9}$$

If we take the visible brane to be one of the three-branes at either $r_i = \omega_i$ with $i = 1, 2$, then it is clear from eq. (4.8) that a large redshift is easily generated without introducing any large parameters in the model. In fact, for y not too large, the warp factor has essentially the same exponential form as in the Randall-Sundrum scenario, i.e.,

$$W(y) \sim \frac{\omega}{L} \exp\left(\frac{y}{L}\right), \tag{4.10}$$

which is essentially a reflection of the fact that the AdS soliton is asymptotically locally AdS. In the present construction, this hierarchy is not an adjustable parameter in the theory, rather it will be fixed implicitly by the relative tension of the branes (and the value of the cosmological constant).

One may hope to find a realization of this new warped brane model or some closely related geometry in string/M-theory. One apparent obstacle would be that our construction involves both three-branes and four-branes. In a given type-II string theory, the dimensions of the different D-brane species in a given string theory always differ by two [33]. However, one could consider working in the type-IIA theory where one finds NS5-branes as well as D4- and D6-branes. This may provide a natural framework to attempt a higher dimensional compactification which provides a close analog of our model based on the AdS soliton.

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A. Delta functions in curved space

In this appendix we discuss the Δ-functions used throughout this paper, and show how they can be calculated. The idea is simple; we need a covariant form for the delta function so as to maintain its familiar properties whenever we are working in a curved space. Since this is an issue of coordinate invariance, the need to modify the delta function prescription also occurs in flat space. It is convenient to consider this case and then make the generalisation to curved space. We will follow the treatment in ref. [34].

In cartesian coordinates in flat, two-dimensional space, the definition of the delta function is

$$\int \mathrm{d}x\mathrm{d}y\, F(x,y)\delta(x-x_0)\delta(y-y_0) = F(x_0,y_0)\,, \tag{A.1}$$

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where $F(x,y)$ is some function defined on the plane. If we transform to polar coordinates, define $H(r,\theta) = F(x(r,\theta), y(r,\theta))$, and assume that neither r nor θ is ignorable at the point $P = (x_0, y_0) = (r_0, \theta_0)$, i.e., that the coordinate transformation is invertible at this point, we get

$$\int \mathrm{d}r\mathrm{d}\theta\, r H(r,\theta)\Delta(r - r_0)\Delta(\theta - \theta_0) = H(r_0, \theta_0)\,, \tag{A.2}$$

which suggests that the Δ-functions should take the form

$$\Delta(r - r_0)\Delta(\theta - \theta_0) = \frac{1}{r}\delta(r - r_0)\delta(\theta - \theta_0)\,. \tag{A.3}$$

In other words, the correct prescription for the Δ-function should involve a term that cancels the coefficients of the metric appearing the measure. Making the obvious generalisation to curved space gives

$$\Delta(\xi - \xi_0) = \frac{1}{\sqrt{G_{\xi\xi}}}\delta(\xi - \xi_0)\,, \tag{A.4}$$

where $G_{\xi\xi}$ is the ξ coefficient of the spacetime metric (assumed diagonal).

The situation is a little more involved if one or more of the coordinates is ignorable at the point P. Consider the case when P is the origin of the flat (r,θ) plane, meaning that $r_0 = x_0 = y_0 = 0$ and θ_0 is ignorable. Then H can only be a function of r, and eq. (A.2) becomes

$$\begin{aligned}\int \mathrm{d}r\mathrm{d}\theta\, r H(r)\delta(x)\delta(y) &= \int \mathrm{d}r\, H(r) \int \mathrm{d}\theta\, r\delta(x)\delta(y) = H(0)\\ &\Rightarrow \int \mathrm{d}\theta\, r\delta(x)\delta(y) = \delta(r)\,.\end{aligned} \tag{A.5}$$

It follows from this equation that $\delta(x)\delta(y)$ cannot be a function of θ, and we are free to write

$$\delta(x)\delta(y) = \Delta(r - r_0)\Delta(\theta - \theta_0) = \frac{\delta(r)}{\int \mathrm{d}\theta\, r} = \frac{\delta(r)}{2\pi r}\,. \tag{A.6}$$

In this case we must cancel not only the metric factor in the measure, but also the integral over the ignorable coordinate. We adopt the same prescription when dealing with ignorable coordinates in curved space. That is,

$$\Delta(\xi^1 - \xi^1_0)\cdots\Delta(\xi^N - \xi^N_0) = \frac{\delta(\xi^1 - \xi^1_0)\cdots\delta(\xi^n - \xi^n_0)}{\int \mathrm{d}\xi^{n+1}\cdots\mathrm{d}\xi^N \sqrt{G_{\xi^1\xi^1}\cdots G_{\xi^N\xi^N}}}\,, \tag{A.7}$$

where $\{\xi^{n+1}, \ldots, \xi^N\}$ are ignorable.

This argument can no doubt be generalised to the case of a non-diagonal metric and put on a firmer mathematical footing by considering the general transformation properties of the Δ-function (for instance, in a D-dimensional space, the full D-dimensional delta function must transform as a relative tensor of weight -1 to ensure that $\int \mathrm{d}^{\mathrm{D}}\xi\, \delta^{(\mathrm{D})}(\xi) = 1$ is a coordinate invariant). However, the heuristic discussion given above describes the basic idea and is sufficient for our purposes.

We conclude with a derivation of the Δ-functions in eq. (3.33), where the metric is that of the AdS soliton, (3.17). Note that we will use the ρ coordinate, defined in eq. (3.30), and that $G_{\rho\rho}G_{\tau\tau} = 1$. The radial positions of the two conical singularities, $\rho = w$ and $\rho = R_1 + R_2 - w$, are such that τ is ignorable. Therefore, we use eq. (A.7):

$$\begin{aligned}\Delta^{(2)}(y - y_3^{(1)}) &= \frac{\delta(\rho - w)}{\int \mathrm{d}\tau \sqrt{G_{\rho\rho}G_{\tau\tau}}} = \frac{1}{l_1}\delta(\rho - w)\,, \\ \Delta^{(2)}(y - y_3^{(2)}) &= \frac{\delta(\rho - [R_1 + R_2 - w])}{\int \mathrm{d}\tau \sqrt{G_{\rho\rho}G_{\tau\tau}}} = \frac{1}{l_2}\delta(\rho - [R_1 + R_2 - w])\,. \end{aligned} \tag{A.8}$$

Similarly, the position of the de-localised three-brane at the interface is such that τ is ignorable. Using eq. (A.7) again gives

$$\Delta^{(2)}(y - y_3^{(3)}) = \frac{\delta(\rho - R_1)}{L_\tau \sqrt{G_{\rho\rho}}}\,, \tag{A.9}$$

where the denominator on the r.h.s. equals l_1 or l_2 depending upon whether one uses τ_1 or, respectively, τ_2 to parametrize the circle at the interface. The four-brane is localised only in ρ, which coordinate is never ignorable. Hence, using eq. (A.4) we find

$$\Delta(y - y_4) = \frac{\delta(\rho - R_1)}{\sqrt{G_{\rho\rho}}}\,. \tag{A.10}$$

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